ELEN 4810 Final Exam

Monday, December 16, 2024, 4:10-7:00 PM. Two sheets of handwritten notes are allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper.

There are a total of 4 questions. Good luck!

Name:

Uni:

1. $\mathcal{Z}$-transform. Consider a rational transfer function

$$H(z) = \frac{z^{-2}}{(1 - \frac{3}{2}z^{-1})(1 - j\frac{1}{2}z^{-1})(1 + j\frac{1}{2}z^{-1})} \tag{1}$$

(a) What are the poles and zeros of H?

(b) Suppose the system is stable. What is the region of convergence (ROC)?

(c) Now suppose the system is causal. What is the region of convergence (ROC)?

(d) Suppose we express $H(z)$ as $H(z) = H_{\min}(z)H_{\mathrm{ap}}(z)$, with $H_{\min}(z)$ minimum phase, and $H_{\mathrm{ap}}(z)$ all-pass. Please sketch the pole-zero diagrams of $H_{\min}$ and H_{ap}, indicating the multiplicity of any repeated poles or zeros.

Answer to Problem 1 [9 points total]:

(a). [2 points] Poles at $\rho = 3/2, j/2, -j/2$. Zeros at $\zeta = 0$ and $\zeta = \infty$ (multiplicity two)

(b). [2 points] Stable $\Rightarrow$ ROC includes the unit circle. The ROC is

$$\tfrac{1}{2} < |z| < \tfrac{3/2}{.} \tag{2}$$

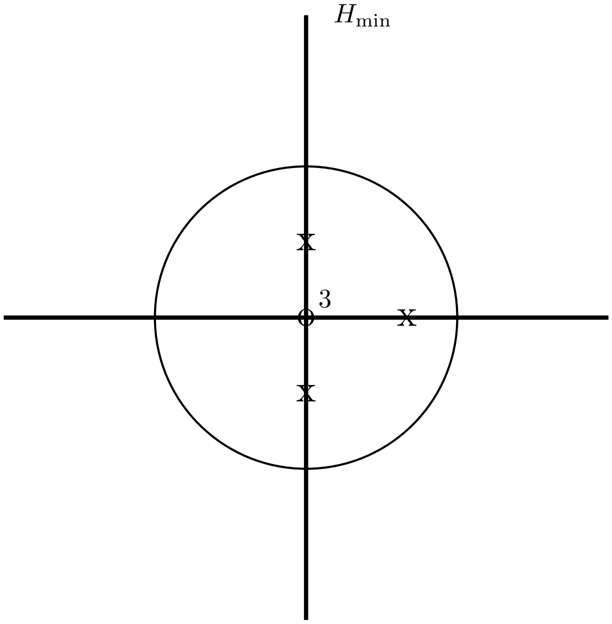
(c). [2 points] Causal $\Rightarrow$ ROC extends outward from largest magnitude pole:

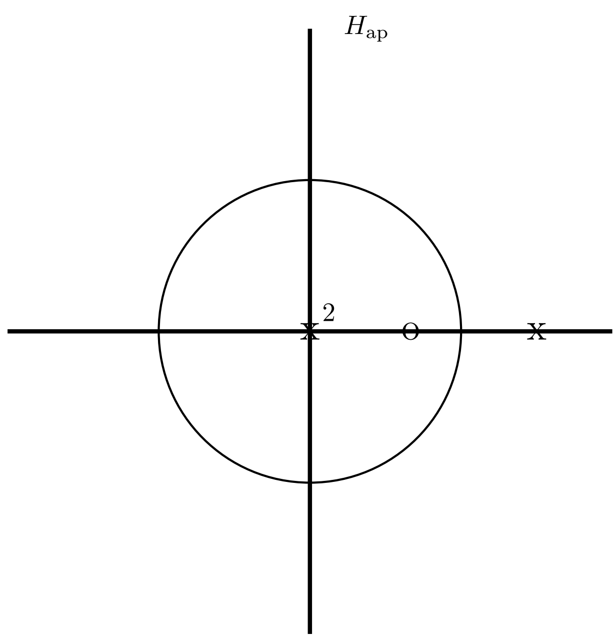
$$|z| > \tfrac{3}{2} \tag{3}$$

(d). [3 points] The system has a pole outside the unit circle ($\rho = \frac{3}{2}$) and two zeros outside the unit circle $\zeta = \infty$ with multiplicity two. Write

$$H(z) = \underbrace{\left[\frac{1}{(\frac{3}{2} - z^{-1})(1 - j\frac{1}{2}z^{-1})(1 + j\frac{1}{2}z^{-1})}\right]}_{\text{minimum phase}} \underbrace{\left[\frac{(\frac{3}{2} - z^{-1})z^{-2}}{(1 - \frac{3}{2}z^{-1})}\right]}_{\text{all pass}} \tag{4}$$

Axes for part (d):

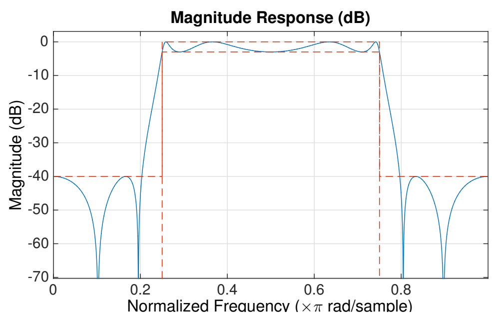




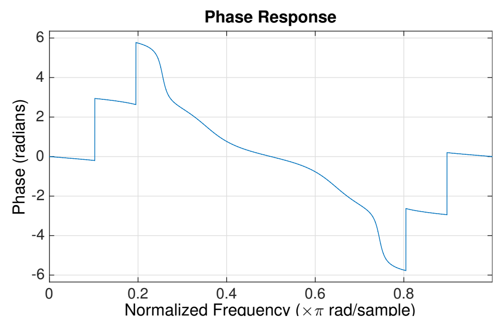
2. Bandpass Filters via Three Approaches. In this problem, we set out with the goal of designing a *bandpass filter* whose magnitude response approximates a target magnitude response

$$|H_{\text{Target}}(e^{j\omega})| = \begin{cases} 1 & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0 & \text{else}, \end{cases} \tag{5}$$

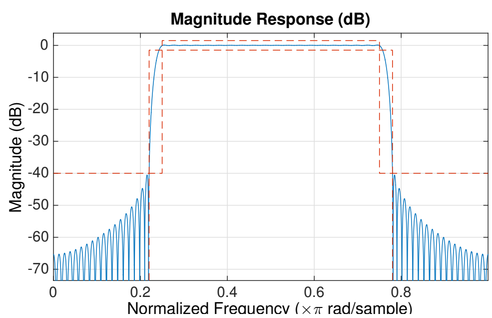
over $-\pi \leq \omega \leq \pi$. We design three different filters, with frequency responses $H_1(e^{j\omega})$, $H_2(e^{j\omega})$, $H_3(e^{j\omega})$. The magnitude and phase response of each are shown below.



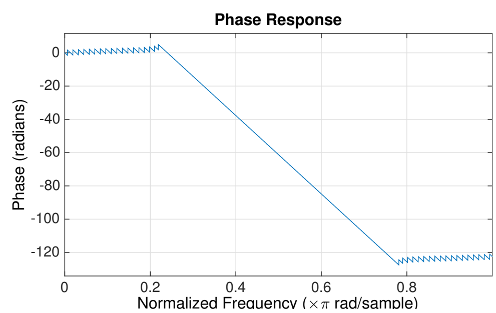
$20\log_{10}|H_1(e^{j\omega})|$



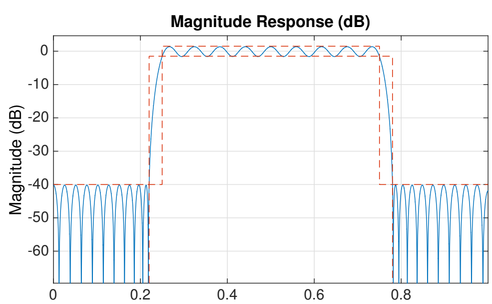
$\angle H_1(e^{j\omega})$



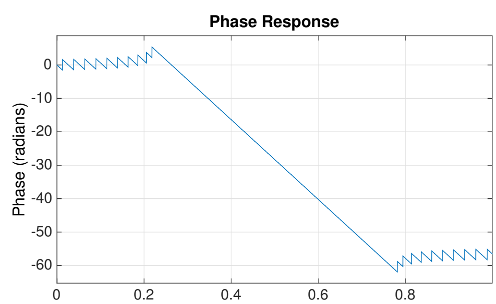
$20\log_{10}|H_2(e^{j\omega})|$



$\angle H_2(e^{j\omega})$



$20\log_{10}|H_3(e^{j\omega})|$



$\angle H_3(e^{j\omega})$

PART 1. How were these filters designed? Consider the following filter design methodologies:

(A) Window. Windowing a target impulse response $h_{\text{Target}}[n]$ with a Kaiser window, to produce a Generalized Linear Phase FIR filter.

(B) Parks-McClellan (L^{∞} optimization). Designing an optimal Type I Generalized Linear Phase FIR filter via the Parks-McClellan algorithm.

(C) Minimum Energy. Designing a length-N FIR filter, by choosing the nonzero portion of the impulse response, $h[0], \ldots, h[N-1]$, to minimize the squared error

$$\int_{\omega=-\pi}^{\pi} \left| H(e^{j\omega}) - H_{\text{target}}(e^{j\omega}) \right|^2 d\omega. \tag{6}$$

(D) IIR Elliptic. Transforming a continuous-time *elliptic filter* $H_a(s)$ to discrete time, by setting $H(z) = H_a(\varphi(z))$, where $\varphi(z)$ is some rational function of z.

(E) IIR Butterworth. Transforming a continuous-time *Butterworth filter* $H_a(s)$ to discrete time, by setting $H(z) = H_a(\varphi(z))$, where $\varphi(z)$ is some rational function of z.

For each of the three designs H_1, H_2, H_3, please indicate which design methodology was used. For full credit, please explain how you arrived at your answers.

PART 2. Advantages and disadvantages. Why might we prefer System 2 (with $H_2(e^{j\omega})$) to System 1 ($H_1(e^{j\omega})$)? Why might we prefer System 1 ($H_1(e^{j\omega})$) to System 3 ($H_3(e^{j\omega})$)?

Answer to Problem 2 [10 points]:

PART 1 [6 points]. Filter 1 is an IIR elliptic filter – it exhibits ripple in both the passband and stopband. Filter 2 is an FIR filter produced by Kaiser windowing. Filter 3 is an FIR filter produced by the Parks-McClellan algorithm. It exhibits generalized linear phase and is equiripple.

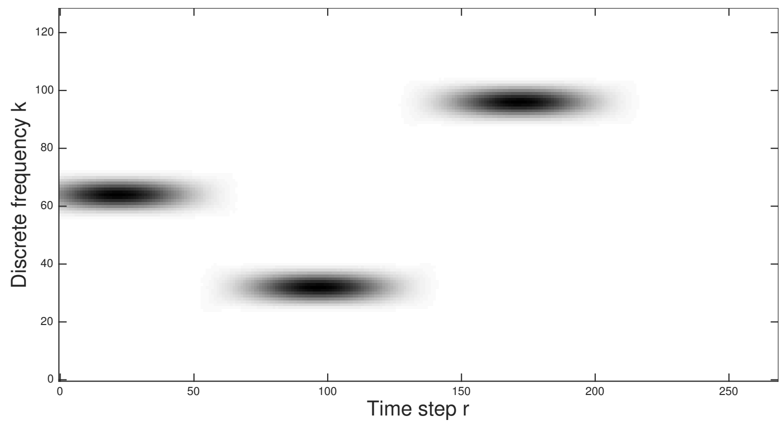
PART 2 [4 points]. The major advantage of System 2 over System 1 is that System 2 exhibits generalized linear phase.

The major advantage of System 1 over System 3 is that System 1 will be more efficiently implementable (lower order).

3. Linear Systems and STFT. Recall the definition of the Short-Time Fourier Transform

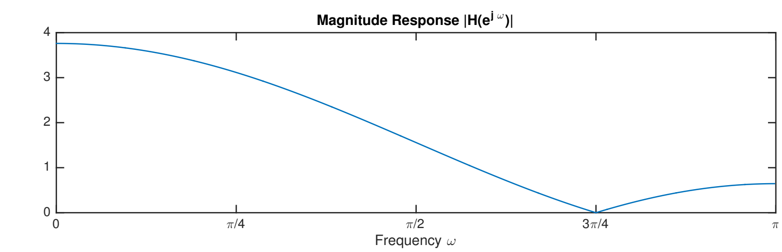
$$X[r,k] = \mathrm{DFT}_N \{w[n]x[n+rR]\}[k]$$

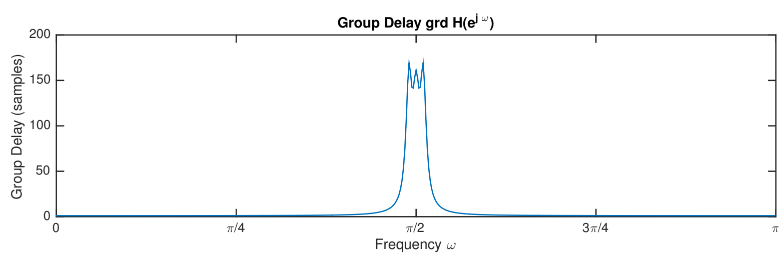
We compute $X[r,k]$ for a signal $x[n]$, with the following parameters: $N = 256$, $R = 2$, and $w[n]$ a Hamming window of length 64. Below, we show the magnitude $|X[r,k]|$ for $r = 0,\ldots,268$ and $k = 0,\ldots 128$.



In the above figure, the three largest values of $|X[r,k]|$ occur at (i) $r = 21$, $k = 64$, (ii) $r = 96$, $k = 32$, and (iii) $r = 171$, $k = 96$. At each of these peaks, $|X[r,k]| = 1$.

The signal $x[n]$ is passed through a linear, time-invariant system with frequency response $H(e^{j\omega})$ to produce an output $y[n]$. The magnitude response $|H(e^{j\omega})|$ and group delay $\mathrm{grd}[H(e^{j\omega})]$ are shown below:

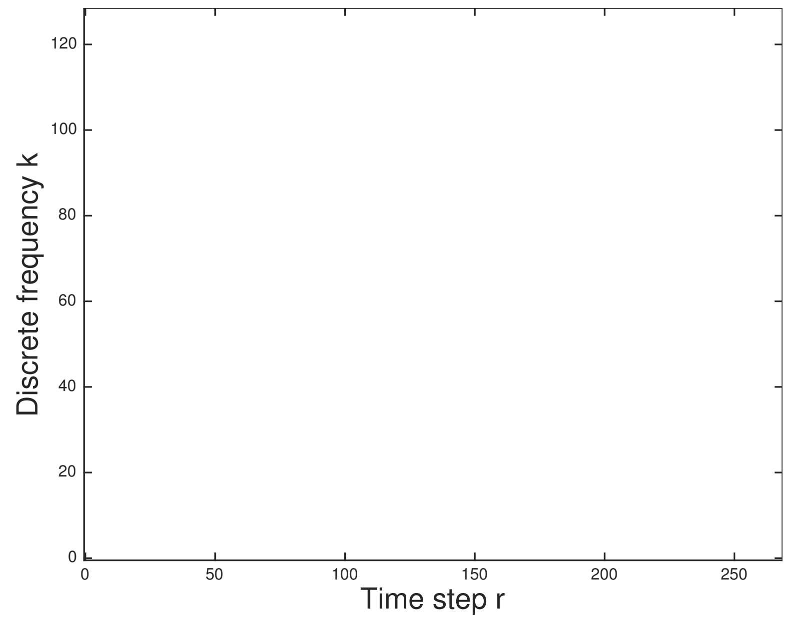




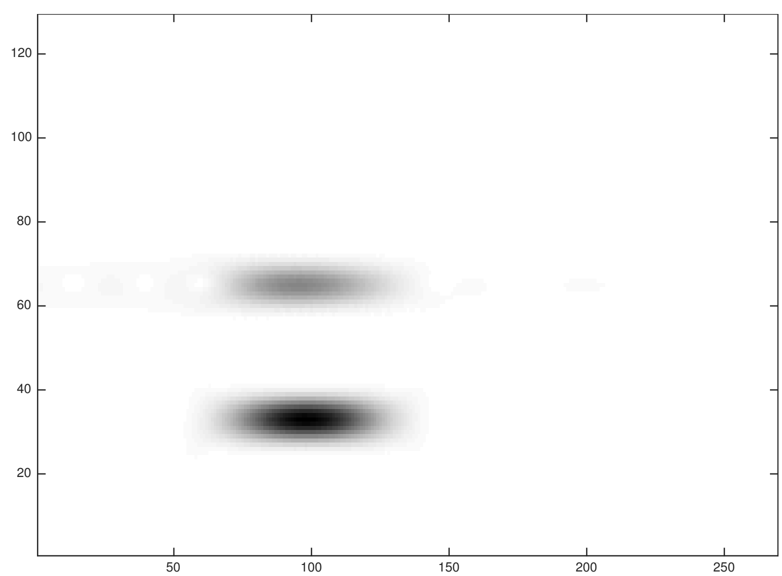
We compute the Short-Time Fourier Transform $Y[r,k]$, with the same parameters. On the axis below, please sketch the spectrogram $|Y[r,k]|$. Please label the approximate location (r, k), and height $|Y[r,k]|$ of any peaks in the spectrogram.

[Please briefly explain the reasoning behind your sketch.]

Answer to Problem 3. [9 points – Please allocate 3 points for each frequency component.] Please sketch your answer below:



Solution: we include the true output of the system below:



The reasoning leading to this answer is as follows: the high frequency component at $r \approx 171$ in the input is centered about discrete frequency $k = 96$. This corresponds to $\omega = 2\pi k/N = 3\pi/4$. The magnitude response is zero at $\omega = 3\pi/4$, and so this component is almost completely attenuated.

The low-frequency component at $r = 96, k = 32$ corresponds to frequency $\omega = 2\pi k/N = \pi/4$. $|H(e^{j\pi/4})| \approx 3$, and so this component will be amplified by a factor of ≈ 3. The group delay at $\pi/4$ is very close to zero, and so this component will occur at roughly $r = 96$ in the output.

Finally, the medium frequency component at $r = 21, k = 64$ corresponds to frequency $\omega = \pi/2$. The magnitude response at this point is roughly 1.5, and so this component will be amplified by roughly 1.5. The group delay is roughly 150 samples, and so this component will be delayed by $n = 150$ samples. Since the spectrogram step is $R = 2$, this corresponds to a delay in $r = n/R$ by roughly 75. This component will occur at $k = 64$ and $r \approx 21 + 75 = 91$ in the output.

[[Note: minor imprecisions in the estimates of these quantities are fine, as long as the reasoning is correct.]]

4. Bilinear Transform. A continuous-time system with transfer function $H_c(s)$ has poles at $s = -\frac{1}{3}, -3$ and zeros at $s = \frac{1}{3}, 3$. We design a causal, stable discrete time system $H(z)$ by applying the bilinear transformation to $H_c(s)$, setting

$$H(z) = H_c\left(\frac{z-1}{z+1}\right)$$

Part (i). Please sketch the poles, zeros and region of convergence of H(z)

Part (ii). Which of the following best describes $H(z)$?

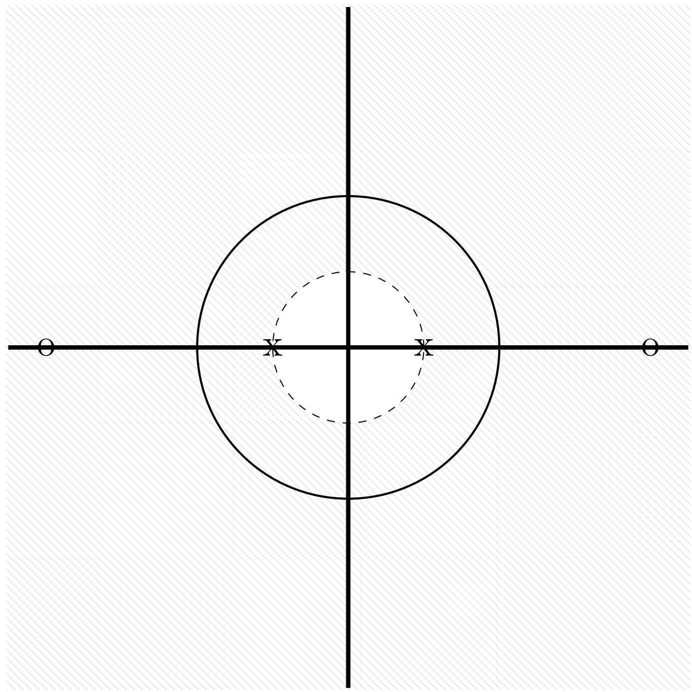
LOWPASS BANDPASS HIGHPASS ALLPASS BANDSTOP MINIMUM PHASE

Part (iii). Now, consider a modified system with transfer function

$$H'(z) = H(z)\left(\frac{1+\frac{1}{4}z^{-2}}{1-\frac{1}{4}z^{-2}}\right).$$

At what frequency or frequencies ω is the magnitude response $|H'(e^{j\omega})|$ *maximized*?

Answer to Problem 4 [8 points]:



Part (i) [3 points]. A pole/zero at location s maps to location

$$z = \frac{s+1}{1-s}. \tag{7}$$

Evaluating at $\pm\frac{1}{3}$, ± 3, we obtain that $H(z)$ has poles at

$$\rho = \frac{-\frac{1}{3}+1}{1--\frac{1}{3}} = -\frac{1}{2} \tag{8}$$

$$\rho = \frac{-3+1}{1--3} = \frac{1}{2} \tag{9}$$

and zeros at

$$\zeta = \frac{\frac{1}{3}+1}{1-\frac{1}{3}} = 2 \tag{10}$$

$$\zeta = \frac{3+1}{1-3} = -2 \tag{11}$$

These poles and zeros are plotted above. Because the system is causal, the ROC extends outward from the largest magnitude pole:

$$\text{ROC} = \{z \mid |z| > \tfrac{1}{2}\} \tag{12}$$

Part (ii) [2 points]. All pass: the poles and zeros occur in conjugate reciprocal pairs.

Part (iii) [3 points]. Because the $H(z)$ is all pass,

$$
\begin{aligned}
|H'(e^{j\omega})| &= \left| \frac{1+\frac{1}{4}z^{-2}}{1-\frac{1}{4}z^{-2}} \right|_{z=e^{j\omega}} &(13)\\
&= \left| \frac{\left(1+j\frac{1}{2}z^{-1}\right)\left(1-j\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \right|_{z=e^{j\omega}} &(14)
\end{aligned}
$$

This system exhibits poles at $\pm\frac{1}{2}$ (corresponding to $\omega = 0, \pi$) and zeros at $\pm j\frac{1}{2}$ (corresponding to $\omega = \frac{\pi}{2}, \frac{3\pi}{2}$.

The magnitude response is maximized when $\omega = 0, \pi$. (For maximum completeness, because of the 2π periodicity of the DTFT, when $\omega = k\pi$ for any $k \in \mathbb{Z}$; we will not deduct points for answers that only list $\omega = 0, \pi$ or $\omega = 0, -\pi$).